CONCEPT OF *ŚŪNYA* IN INDIAN ANTIQUITY

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Abstract

The genesis of "zero" as a number, that even a child so casually uses today, is a long and involved one. A substantial number of persons concerned with the history of its evolution, today accept that the number "zero", in its true potential, as we use it in our present day mathematics has its root, conceptually as well as etymologically, in the word "*Śūnya*" of Indian antiquity, and it was introduced in India by the Hindu mathematicians. This eventually became a numeral for mathematical expression for "nothing", and via the Arabs, went to Europe. The time frame of its origin in Indian Antiquity is still hotly debated. One recent suggestion, from some astronomical calculations is that, it probably appeared in 458 CE as found in the Jaina cosmological text *Lokavibhāga*(meaning, *The Parts of the Universe*). Furthermore, some recent works even try to suggest that a trace of the concept, if not in total operational perspective, might have a Greek origin that traveled to India during the Greek invasion of the northern part of the country in the pre-Mauryan period.

In this article we would like to discuss the available references to the concept of "*Śūnya*" or its numerous synonyms in its broader social and philosophical contexts as was used in Indian antiquity, which eventually paved the path for the evolution of the corresponding mathematical concept. From the works on Vedic prosody by Piṅgala (*Chandaḥsūtra*) [3rd Century BC] to the concept of "*lopa*" in the grammarian Pāṇini (*Aṣṭādhyāyī*) (sometime between 700-400 BCE, by some modern estimates) it appears very likely that the thread of rich philosophical and socio-academic ambiences of Indian antiquity was quite pregnant with the immensity of the concept of "*Śūnya*"— a dichotomy as well as a simultaneity between nothing and everything, the "zero" of void and that of an all pervading "fathomless" infinite.

1 Introduction

"History is, by definition, the period for which we have written sources. In this sense, the history of Western civilization begins roughly about 3000 BC. In its 5000-year history, different nations have, over time, occupied center stage by virtue of intense displays of complex activity"— So observed Leo Depuydt,[13] a noted historian of Antiquity. Going by such a 'definition', one may be prompted to reach a conclusion as follows: "Our current numbering system is Indian in origin and took shape during the period of cultural and intellectual splendour that took place along the valley of the Ganges from the mid III century to the mid VI century AD. During this period, the Gupta dynasty reigned the region."[15]

However, when the subject under scrutiny is related to the genesis of mathematical knowledge in Indian Antiquity, particularly in the pre-Christian era, perhaps dating back to several millennia further, the aforesaid 'definition' cannot be of much help owing to the complete unavailability of written documents pertaining to this period. However, from the huge collection of available documents of later dates[1], beginning from the early Christian era, where one may find regular references to ancient works of amazingly lofty philosophical depth of thought, mingled with occasional flashes of extraordinary scientific sophistication, it goes without saying that the remote antiquity in India cradled a civilization of a very high order. As Sri Aurobindo observes "Power of the ancient Indian spirit was a strong intellectuality, at once austere and rich, robust and minute, powerful and delicate, massive in principle and curious in details".[3] It is with this background of the social context and ambience of the Indian Antiquity in mind, that one must try to judge the possible potential mathematical developments, as testified passively by various apparently non mathematical resources, in case of the absence of direct mathematical testimony. In this article we shall try to throw some light in this direction, with an intention to illuminate some of the mystic and shadowy corners of this great culture.

"*Mathematics is by definition universal; one expects it to be essentially the same everywhere. However, that does not mean that the modes of expression may not differ, leaving opportunities for detecting historical links. Still, the burden of proof in deriving historical links from similarities is on the whole more onerous in the case of mathematics than it is with just about any other type of human activity*".[13]

All that said and done, in a recent scholarly book by Kaplan[34] one may find the following remark, "*Perhaps the world would be better and the past more attractive if some dead Indian*

[1]The mathematical heritage of the Indian subcontinent has long been recognized to be extremely rich. Hundreds of thousands of manuscripts in India and elsewhere, mostly written in Sanskrit, attest to this tradition.[48]

had devised the hollow circle for zero rather than some dead Greeks (though I can't see why, especially since the concept matters more than its marker, and the concept, as we have seen, goes back to some dead Babylonians)."

The Babylonian place-value system was sexagesimal, or based on the number 60. Some historians believe that its influence in India is found in the later astronomical works,(though this hypothesis is not widely accepted) but not in early India[42]. And the Babylonian concept of zero, found inscribed in their clay tablets, denoted by two slanted wedge signs, as one may find in Labat's *Manuel D'Épigraphie Akkadienne* Paris(1948) or Kugler's *Babylonische Mondrechnung*, Tafel II (and its transcription, pp.34), (1900) were only placeholders, sign of separation, which was never used alone as a number in its own right. There is no zero in the scientific texts of first Babylonian Dynasty, and the figure is hardly attested in any texts prior to third century BCE[25](pp. 153). Mathematicians seem to have used the zero in the intermediate position of a number only, but the astronomers used it not only in the middle but also in the final and initial positions of numerical expressions[2]. Furthermore, The double wedge symbol for zero had the meaning of "void", but it does not appear to have been imagined as "nothing". Ifrah [25] cites certain examples of mathematical tablets from Susa depicting cases where numerical calculations were being done. When the outcome of a calculation resulted in a clearcut 'zero', the scribe seemed to be at a loss, and instead of inscribing their slanted double wedge there, either left the place vacant, or took recourse to language like "*the grain is finished*." This allows us to conclude, following Ifrah[25], that "*the concept of 'void' and 'nothing' both existed(*in Babylonian mathematics*). But they were not yet seen as synonyms*."

However, we are fortunate to have ample inscriptions on clay tablets bearing, among other things, monetary documents, testifying the mathematical knowledge and skill of that period, from as early as pre-Sargonic era (the first half of the third millennium BCE), that these ancient Sumerians have bequeathed to us. Pity that no such document of early Indian Antiquity is extant. Before we begin to interpret the thoughts and history of this bygone era, we humbly recall Baruch Spinoza (1632-1677) : '*Not to ridicule, not to mourn, never to detest, but try to understand*'[3].

[2]Oscar Neugebauer (Zur Entstehung des Sexagesimalsystems, Göttingen, 1927) has shown such examples in an astronomical tablet of Selucid period of Babylon. There is a hypothesis that this might have been a copy of some earlier documents (H. Hunger, Spätbabylonische Texte aus Uruk, Berlin, 1976). But these are only suppositions. Only further archaeological discoveries can provide definite proof.[25]

[3]"Sedulo curavi, humanas actiones non redere, non lugere, neque detestari, sed intelligere." The translation given here, is credited to Sridharan[61].

2 Historiographic Problems

"Some of the chief historiographic difficulties surrounding Indian mathematics traditions include the fact that, the European scholars who encountered Indian mathematical texts in the eighteenth and nineteenth centuries were often completely at sea concerning the ages of the text, their interrelationship and even their identities. The sheer number of such works and the uncertainty surrounding the most basic chronology of the Sanskrit literature gave rise to great confusion, much of which survives to this day in discussions on Indian mathematics.[48]

Though some pioneering contributions of the Indian mathematicians of ancient period is generally acknowledged, there are reasons to believe that many famous historians of mathematics, while trying to find their way through this bewildering maze of extant Sanskrit texts towards establishing the exact nature and content of the Indian contribution, with a special focus on their period of inception, "*found what they expected to find, or perhaps even what they hoped to find, rather than to realize what was so clear in front of them.*"[43]

As a matter of fact, a good volume of mathematical works available for scholarly scrutiny, are embedded in the contexts of medieval Indian astronomy and astrology, not naturally familiar to the European scholars. Furthermore, the presentation of these texts is usually found to be in highly compressed Sanskrit verses, frequently cryptic to the point of being obscure, and that too, not always in the present form of Sanskrit, but in what is called the Vedic Sanskrit, alien in appearance to a non specialist. '*Further, many texts, even on technical subject like* jyotiṣa, *were ascribed to the revelations of gods or legendary sages. This attribution expunged the historical contexts of the works to stress the divine importance of their content. Similarly, even historical human authors frequently omitted biographical information and other contextual details as irrelevant and unnecessary. This sometimes makes it difficult to distinguish between human and allegedly divine authors*'.[48] All these difficulties took their toll on the early surveys of Indian sources, 'which often tended to portray the Indian mathematical tradition as a record of discoveries or contributions, classified according to modern mathematical categories and important in proportion to their originality or priority'. The context for understanding Indian mathematics in its own right, as an integral part of the social and cultural ambience of Indian Antiquity, was generally neglected. There was yet another menace. '*The historiography of science in India has long been co-opted for political purposes. Most notoriously, some nineteenth century colonial officials disparaged local intellectual traditions, which they termed "native learning", in order to justify the westernized education for future colonial servants.*'[48] Adding to the injury is the fact that much of the desired data is simply unavailable from India's historical records, as far as the present state of information in this regard is concerned. Plofker[48] in a seminal work points out, "*it is only around the middle of the first millennium CE that one finds the first surviving complete Sanskrit text in the medieval tradition of mathematical astronomy*"— quite a frustrating state of affairs, more so, as it is known beyond doubt, that several surprisingly advanced civilizations with rich philosophical, linguistic and scientific culture flourished in India, predating the Christian era by millennia. This unfortunate information vacuum has given rise to an almost chaotic condition with myths and controversies, claims and counter claims, disputed 'proof's and fallacious presumptions. In one extreme we find the conscious shaping up of the history according to some pre-conceived 'Eurocentric' bias amounting to the doctrine aptly summarized by R. Rashed as "*Classical science is European and its origins are directly traceable to Greek philosophy and science*" and on the other we have the "*bandwagon of nationalist historiographers, who are obsessed with establishing the Indian priority in every aspect in the history of ideas [55]*", often putting forward extravagant claims for the autonomy or antiquity of their scientific tradition.[4] They present some evidences as irrefutable proof which the other group promptly rejects as forgeries. However, many of the generally accepted conclusions are nonetheless not definitely proved, and many revisionists or minority views have achieved widespread popularity. As a result, a shift in the focus of opinions regarding certain claims and counter-claims in this very alive and vibrant area of study has become quite commonplace. As for example, one may cite the issue of originality of the decimal arithmetic of Āryabhaṭa, which is nowadays almost universally accepted to be independent of the Greek sources[5], though it was hotly debated even a few years back.[48]. However, the crux of the historiographic enigma regarding the mathematical sciences in Indian Antiquity is ultimately poised at the questions of when, how and by whom these texts were composed, and *these questions still have no universally accepted answer in most of the cases.*[6]'

[4]Quoting R.C. Gupta[20],"*in respect of Vedas and Vedic Corpus, there seems to be no moderation in regard to highly fantastic claims of advanced scientific achievements. These include the proof of famous Fermat's last equation and the Einsteinian mass energy equation....*"

[5]Roger Billard[5] demonstrated that Āryabhaṭa has made his own independent observations and his astronomical constants and parameters are not borrowed from the Greeks.

[6]A good example is the uncertainty of time period regarding the inception and flourish of Vedic religion. "*S. Kak[30], a noted authority on Vedic Astronomy, states in a recent work that the time period for the Vedic religion stretches back potentially as far as 8000 BC and definitely 4000 BC. Whereas G. Joseph states 1500 BC as the forming of the Hindu civilization........R. Gupta in his paper on the problem of ancient Indian chronology shows that dates from 26000-200 BC have been suggested for the Vedic period. Having consulted many sources I am confident at placing the period of the Vedas (and Vedangas) at around 1900-1000 BC*"[47]

3 Enigma of Migration

While talking about 'Indian Antiquity', which refers to the known period of history up to around 6th century A.D., it is important to realize that, the group of inhabitants it refers to, particularly in the remote antiquity is still somewhat shrouded in mystery. Who were these people, supposedly so preoccupied with the pursuit of timeless spiritual knowledge, rather than the recording of mundane events of daily social life? This could have been called their '*history*'−a term, which so rarely, if at all, can be ascribed to any available literature belonging to the vast corpus of Sanskrit texts. Where did they come from, or did they originate here, in India? However, it must be kept in mind that, while referring to India in antiquity, as per common norms, one takes into account, the entire geographical locus of the present day nations of Pakistan, Nepal, Bangladesh, Sri Lanka, along with present day India. Analysing the available references from their spoken language, an archaic form of Sanskrit, in which the earliest known texts, called The Vedas were composed, a school of linguists categories them as Proto-Indo-European, which refers to various Indo-European themes and motifs found common in this culture with that of some ancient European ones. However, it is observed that, the origin and diffusion of the common ancestral Indo-European cultures are still quite problematic. [48] It is generally proposed that a group of peoples, originally belonging to that culture, once came down to the Indian soil, through the high altitude mountain passes of Himalaya and eventually got settled there. This theory of the origin of Vedic Aryans, is usually referred to as the *Arian Invasion Theory*, or as the *Arian Immigration theory*, as some might prefer to call. However, there are numerous difficulties with most of the features of this hypothesis. [48]

Contesting this hypothesis, one has the so called *Indigenous Aryan Theory*, where one has no need to link the Vedas and their language to a hypothetical Indo-European expansion over the Afgan highlands. This theory claims that the prehistoric agrarian-urban culture of Indus Valley civilization, centered at Harappa, identified during an excavation in 1921-22 and Mohenjo-Daro, found around the same time, which flourished during mid-third millennium BCE near the Indus river around Sind and Punjab, was the Vedic Culture in reality. Comparatively recent excavations, in nearby locations have revealed cites like Mehrgarh,(excavation started in 1974, under a French archaeologist, J.F. Jarrige, near Bolan pass which is about 150 km from the Pakistani city of Queta) which is believed to have cradled an urban civilization that dates back to 7000 BCE. It appears that one can claim with confidence that the extraordinary advancements found in various facets of the Harappan civilization like agriculture, architecture, manufacture and trade was not a sudden development, but a gradual culmination of these various other folk cultures and urban civilizations, predating it by quite a few millennia.

Yet another hypothesis, called *Out of India*, not conforming much to the linguistic angle, suggests that instead of a few Indo-Europeans coming down to the subcontinent through the mountain passes, perhaps most Indo-Europeans had gone out of it! However, this theory also has several other weak points, as the evidences of Vedic Sanskrit culture, mostly textual, does not match with that of the Indus Valley and related ones. These are mostly archaeological in nature, and hence difficult to compare, given that, in spite of numerous claims, a large collection of the Harappan epigraphs are still undeciphered[20]. Indeed, even this much is not known with certainty, whether they form a part of some ancient script or are merely nonlinguistic signs. However, early Vedic hymns do not refer to cities or wheat, well known in the Indus culture. On the other hand, archaeological excavations did not reveal remains of some very common characteristic Indo-European items like horses or chariots. Though analysing the evens and odds of these hypotheses, the *Arian Immigration theory* is presently considered to be the standard accepted narrative of the origin of the Vedic Civilization, one must categorically point out that, there is little definite evidence concerning various aspects of this theory. [48] It is also worth mentioning that, it is not known whether the ancient civilizations of Mohenjo-Daro and Harappa still existed when India was 'invaded' by the Aryans, or whether their writing had developed during this period[7].

4 Early Vedic Period

[48] The sacred Vedas, whose name has the same root as that of the word *vidyā* which literally means "knowledge",[8] are often considered as the fountainhead of learning, preserving the knowledge and doctrines of the remote antiquity. The corpus of "Vedic Texts" includes the four *saṁhitās*, the oldest layer (c. 1000 BCE) of Vedic literature. They are collections of hymns and rituals — namely the *Ṛgveda, Yajurveda, Sāmaveda* and *Atharvaveda*, of which the *Ṛgveda* is the oldest. The next phase of Vedic literature, consisting of the commentaries on *saṁhitās* are found in the *Brāhmaṇas* (c.800 BCE) and *Āraṇyakas*(c. 700 BCE), as well as explanatory and philosophical works in *Upaniṣads*(c. 600-500 BCE)[9]. One can find some

[7] It is almost certain that when the Aryans arrived in India they brought no form of writing with them.....Their intellectual and spiritual leaders would certainly have had a "knowledge of the great religious poems;but it seems that their literature was written at a later date, and then the literate men would doubtless have preferred to keep the oral tradition going as long as possible to perpetuate their prestige and their privilege"[M Cohen, 1958][25]

[8] both the words come from the same Sanskrit root $\sqrt{Vid}$, that means "to know"

[9] The approximate dates mentioned here are taken from Joseph[29], who however puts in clear terms the state of affairs regarding the uncertainty in predicting time period of early Indian texts. '*The dates given here are rough and conservative estimates of when the first versions of the texts were recorded. It is very likely that before they were written, an earlier oral tradition kept the contents alive. Copying old texts was a common pursuit of the Indian scholar and student, sanctioned by religion and custom. It is therefore*

references to the study of mathematics as a branch of *aparāvidyā*, the worldly knowledge, in the *Muṇḍaka Upaniṣad* and *Chāndogya Upaniṣad*. In the first millennium BCE, at around the end of the *Brāhmaṇas* phase of the Vedic age, the division of learning included not only the Vedic texts themselves but also the six auxiliary limbs or *ṣaḍaṅgas*, essential for proper understanding of the Vedic texts and rituals. *Vedāṅgas* i.e.,"limbs" or supporting disciplines of the Vedas, *which were composed due to a conscious and meticulous effort towards preservation of the meaning and external forms of the mantras of saṁhitās.*[16] These are,

śikṣā i.e., phonetics, dealing with the description of vowels and consonants towards ensuring the most correct reading of the Vedas and unfolding its inner truth;

kalpa i.e.,ritual practice, which specified the details of the various rites;

vyākaraṇa i.e., grammar, important for correct understanding of the exact meaning of words through perfect utterance;

nirukta i.e., etymology, as complementary to the grammar, enabling one to discover the primary deity of a stanza of hymns which may bear the characteristic marks of more than one deity, thus helping to perform the sacrifice with perfection;

chandas i.e., prosody or poetic metrics, which ensured the proper preservation and comprehension of the archaic verses of the hymns; and

jyotiṣa i.e., astronomy and calendrics, which determined the proper times for performance of the rites.

In *Pāṇinīya śikṣa* we have the following verse:

Chandaḥ pādantu vedasya hastau kalpo'tha paṭhyate
jyotiṣāmayanaṁ cakṣurniruktaṁ śrotamucyate

Chandas are the feet of the Vedas, Kalpa the hands,
Astronomy the eyes and Nirukta the ears.

All these *Vedāṅgas* may be divided into two groups, explanatory and customary. *Kalpa* and *jyotiṣa* fall into the second group while the others belong to the first. The Vedic texts are generally known as *śruti*, "heard" via divine revelation, hence *apouruṣeya*; the limbs of the Vedas, on the other hand is called *smṛiti*, remembered from human tradition. In the post-Vedic era of what is known as Classical Sanskrit, one finds an enhanced form of *jyotiṣa* that incorporated not only astrology but also computational methods in general, known as *gaṇita*, the science of calculation, i.e., mathematics. The exact sciences and most other branches of *smṛiti* learning were called *śāstras*, i.e., "teachings". In the *Vedāṅga jyotiṣa* one finds the mention of the supreme importance of mathematics depicted in the following verse:

important not to depend on evidence of the date of copies of mathematical texts in assessing the true age of a particular method or technique.'

yathā śikhā mayūrāṇāṁ nāgānāṁ maṇayo yathā
tadvadvedāṅgaśāstrāṇāṁ gaṇitaṁ mūrdhani sthitam

Like the crest of a peacock, like the gem on the head of a snake,
so is mathematics at the peak of all knowledge.

However, though the importance of mathematics is so ceremoniously pointed out at such an early stage of Indian Antiquity, unfortunately no exclusively mathematical work of the remote part of this period is extant, except perhaps, the *Śulbasūtras*[10]. Composed in Vedic Sanskrit, the works of this period are primarily religious in content, 'as it was believed that True knowledge of whatever sort was necessarily a part of the fundamental truth of the Vedas', but they tacitly embody a significant amount of mathematical knowledge, chiefly astronomical in classification, which was perhaps necessary to determine the exact timing for the rituals and rites to be performed. Again, the accurate construction of the altars (*vedi*) and fire-places (*agni-kuṇḍa*) for sacrificial rites was necessary and perhaps that paved the path towards the development of geometry[11] that comes to us codified in the nine manuals, called *Śulbasūtras*, associated with the *Yajurveda*. Four of them, *Baudhāyana, Mānava, Āpastamba, Kātyāyana,* [12] form separate treatise, rest are chapters or parts of chapters of the corresponding *Śrautasūtras*[2]. These are the most extant mathematical document of Indian Antiquity known to us, as of now, the earliest one (*Baudhāyana*) among them being placed around c. 800 BCE, justified through its pre-Paninian linguistic character, while the latest one(Kātyāyana) being somewhere around 200 BCE[39][13]. However, there are strong evidences favouring the fact that the knowledge of *śulba* mathematics dates back to the *Saṁhitās* and *Brāhmaṇas*, and the *śulba* authors themselves '*emphasise that they were merely stating facts*' known to the earlier seers.[16] Regarding the tacit presence of mathematical concepts in religious texts, for example decimal nomenclature of numbers of astonishingly high order, not in figures of course, but with the names of the numbers, numerous examples can be found in many treatise dating back to the *Saṁhitās*— the earliest phase of Vedic literature,[18],[16], clearly predating 1000 BCE, even by moderate estimates.[42]

In the early Vedic society, veneration of Sanskrit put it as a sacred speech, *Devabhāṣā*, '*whose divinely revealed texts were meant to be recited, heard and memorized collectively rather than transmitted in writing*'.[48] Naturally it gave rise to the necessity that such

[10]The word *sulba* is derived from the root $\sqrt{\textit{śulb}}$, meaning '*to measure*', whence the term *śulba* stands for measuring cord.

[11]One can refer to the ground breaking work of Sarasvati Amma[2] in this regard. A. Seidenberg[58] pointed out that the Vedic *Taittiriya Saṁhitā* contains the algebraic, geometric and computational aspects of the so-called Pythagoras theorem.

[12]It is suggested that they were Vedic priests and skilled craftsmen.[43]

[13]However, Sarasvati Amma put it between 5th to 8th century BCE.[2]

texts be composed in formats that could be easily memorized. It was done in two modes, either through condensed prose aphorisms, called *sūtras*, or in verse form, particularly in the classical period, beginning in the late first millennium BCE. Along with the passage of time, the second form of presentation brought in numerous synonyms of a single term, for example a number name, coined from different socio-cultural perspectives[14] and invoking 'poetic license' these were used freely, presumably due to the metrical need of the form of composition. Naturally, the spirit of putting preference to the ease of memorization, sometimes came into the way of ease of understanding. As a result, most of the treatises are found to be supplemented, at a later period of time, by one or more prose commentaries or notes, called (*bhāṣya*) or *ṭikā* respectively. Often more than one in number, these works were composed by different authors writing from time to time on the same original text, who got wise either to the need of preservation and proper interpretation of the past knowledge or to improve upon the previously available version by including in it their own reflections and interpretations, thanks to which we have extremely useful information about the nature and content of the original texts, most of which are not extant.

5 *Śūnya* in Various Forms of Perception

In ancient India, the usage of the term *śūnya* had a much broader significance as is reveled from various extant texts of the Indian antiquity as opposed to the much narrower literally translated present day meaning attributed to it through the term "*zero*". A study of this subject leaves one with a sense of wonder at the awesome depth and breadth of ancient Indian thought.

[14]For a comprehensive list of various terms equivalent to *śūnya* used to denote 'zero' in Sanskrit literature, one may refer to Gupta[19] or Sarma[52]

5.1 *Śūnya* in Philosophical Context

In various perspectives, the Sanskrit term *śūnya* does not only convey the sense of void or mere nothingness, but also reveals that of the perfect fullness (i.e., *pūrṇa*)[15], a divine infinititude (i.e., *ananta*) through its association with the metaphysical idea of 'ether', the vast expanse of limitless space (found in a wide variety of terms like *kha, gagana, ākāśa, ambara, vyoma, antarikṣa*), and sometimes a receptive vacuum,'*the unformed and the essence of all that is uncreated*', like what happens after the destruction of the universe (i.e.,*pralaya*) by *Śiva*—the eternal destroyer, a vacuum pregnant with the immensity of creation of a new infinite universe (i.e.,*sṛṣṭi*), created out of the *śūnya* by *Śiva*—the eternal creator. '*The vacuum is truly a living void, pulsating in endless rhythms of destruction and creation*[16]', the duality between the minutest of the minute and the greatest of the great, the *Brahman*, perceived as *aṇor aṁīyān mahato mahīyān*[17].

Is the presence of nothing i.e., śūnya *(reflecting non-existence), different from the absence (akin to voidness) of something or anything (reflecting non-availability)*?[28] The Sanskrit word *śūnya* [18] is probably derived from *śūnā*, [as *śūnā* + *yat*], which is the past participle of $\sqrt{}$*śvi*, which means "to swell" or," to grow", and therefrom by semantic extension, "hollow". In *Ṛgveda*, one may find another meaning : " the sense of lack or deficiency"[19]. '*It is possible that the two different words were fused to give* śūnya *a single sense of absence or emptiness with the potential for growth*'[28], an womb like hollow, ready to swell. Its conceptual derivative, *śūnyatā*, the state of void, is described in the *Nāsadīya Sūkta* of *Ṛgveda*[20]. According to the teaching of the*Tantra*, both *ātman* and Śiva are to be conceived

[15]One finds a typical example in the following invocation (*śāntipāṭha*) of the *Upaniṣads*:

om pūrṇam adaḥ pūrṇam idam pūrṇat pūrṇam udacyate
pūrṇasya pūrṇam ādāya pūrṇam eva avaśiṣyate

Om, the infinite fullness is that Brahman. The same infinite fullness is this Brahman.
The same infinite fullness springs from (is hidden) in the finite universe. When this same infinite fullness is taken away from the infinite fullness, still what remains is infinite fullness.[26]

Gupta[19] observes that, mathematically the second half of this invocation implies $x - x = 0$, a statement which is satisfied by $x = 0$ as well.

[16]Physicist Wheeler says that "Empty space is the place of most violent Physics. "[26]

[17]*Kaṭhopaniṣad 1.2.20*[53]

[18]*Amarakoṣa*, a Sanskrit lexicon written by *Amarasiṁha* around 5th century CE, interprets it as *śūnyṃ tu vaśikaṁ tucchariktake* , which means, zero is also called *vaśikaṁ* i.e., void or empty, *tuccha* i.e, trifling, *riktaka* i.e., emptied or devoid of. [38]

[19]In *Ṛgveda Saṁhitā* the word *śūnam* (3.33.13) is found[12], and though the word finally means a swollen state or hollowness, it has been taken to imply lack, want or adherence[38]

[20]*nāsad āsīn no sad āsīt tadānīṃ nāsīd rajo no vyomā paro yat* meaning,'*Existence was not there, nor non-existence. The world was not, the sky beyond was neither.* '[26]

as voids, '*undifferentiated formless entity*'[21], in the first one (*ātman*) the divine qualities of Śiva have not flourished yet and the objective of 'this void' (*ātman*) is to attain the 'other void' known as Śiva.[8] In this context, one may also refer to the *Mādhyamika* school of the *Mahāyāna* Buddhist's doctrine of devoidness, (*śūnyavāda* i.e., nihilism, of the philosopher Nāgārjuna) who argued[22] in favour of the spiritual practice of emptying the mind of all impressions. According to this doctrine, the highest form of knowledge, *prajñāpāramitā*, is the perception of everything phenomenal as *śūnya*, the pure void. Complete *śūnyatā* is the same as *nirvāna*, which '*encompasses the end and the beginning*'. But how to attain this state of *śūnya*? According to *Nāgasena*, a philosopher of the same school, it is to be done by stepwise negating (subtracting or taking away) all the parts of a thing. It soon paved the path for the corresponding mathematical development of thought, particularly among the Buddhist school— "just as emptiness of space is a necessary condition for the appearance of any object, the number zero being no number at all, is the condition for existence of all numbers".[28]

5.2 *Śūnya* in Mathematical Context

To discuss various facets of the mathematical *śūnya* in Indian Antiquity, following perspectives are of paramount importance:

(i) to trace the concept of *śūnya*, within the frame of a place-value system;

(ii) the symbol(s) used for *śūnya* and

(iii) the fundamental mathematical operations with the digit *śūnya*, considered as a digit in its own right.

The first pursuit, an involved one, will be done at length separately in the next section.

Symbol(s) used for *śūnya*

As far as the symbol for *śūnya* in early phase of Indian antiquity is concerned, if there was any, and we shall argue in the later sections that there is every reason to believe, not parochially, that there was, in all probability, some such symbol in vogue, [23] unfortunately no direct evidence such as inscriptions or written documents favouring this argument is extant. So here again, one has to take recourse to the indirect analysis gingerly, via the socio-cultural root, in search of possible lingual mention of any such symbol. The earliest symbol for *śūnya* in India perhaps was a bold dot(*bindu*[24]), referred to as *śūnya-bindu* in the

[21] *Niṣkala Siva*

[22] He used a form of logic, *catuṣkoṭi*, the famous *tetralemma*[35], claiming the four states "*śūnya*, non-*śūnya*, both or neither" to be considered as both mutually exclusive and jointly exhaustive.

[23] refer to section 8.1

[24] It is found in *Pañchasiddhāntikā* of the master Astronomer *Varāhamihira* (505 CE), though by name, not by symbols[10]. An earlier reference, in language, as *ṣaṭ binduyutāni*, and also as *ṣaṭ khayutāni*, meaning, 'six

extant literatures composed around the early centuries of the Christian era. The association between the concept of zero and its symbol is found in a literary work (*Vāsavadattā* by the celebrated Sanskrit fictionist *Subandhu*, (ca. 400 CE[28], however B.B. Datta[10], Kaplan[34] place it in 620 CE) :

"*The stars shone forth, like zero dots (* śūnya-bindavaḥ*)— scattered in the sky as if on the blue rug, the Creator reckoned the total with a bit of the moon for the chalk*"(translation as in[28]).

This idea was dormant in the Vedic literatures as well. The word *kṣudra* (meaning, very small) is found in the *Atharvaveda* [25] (XIX. 22.6 [51]) in comparison with *ṛca*, the positive and *ạnrca*, the negative numbers.[10] It seems quite natural that the word numerals equivalent to *śūnya* like *randhra* (meaning a hole) found in *Amarkośa* or its synonym *chidra*[26] (meaning a puncture mark), gradually got enfigured into the round symbol of *śūnya*, a solid circular dot to begin with, and then at a later stage only the peripheral circle, perhaps to save time required for darkening its interior.

The *Bakṣālī* Manuscript[27]—the most ancient mathematical document of the Indian Antiquity known so far, was found in 1881 at a place called Bakṣālī, about eighty km. north-east of Peshawar (now in Pakistan), by a farmer who dug it up while cultivating a land. The untitled manuscript, written on birch-bark, is composed in *Gāthā* language (a modified form of *prākṛta*) and the scripts used is recognised as an earlier type of *Śāradā*, once used in Kashmir region of Indian subcontinent. In the seventy fragmentary leaves of the document, which is now extant, one finds numerous intricate calculations involving decimal place-value notation of numerals, including a solid dot [28] for the digit *śūnya*.[11] The time and purpose of composition of this text is widely debated. The work has never been quoted or mentioned in any other works and might even be just an exercise compiled by a scribe.[29] Estimates of

with zero', representing 60, (Incidentally, the most common parlance for mathematical *śūnya* in ancient India was *kha*) can be seen in *Yavanajātaka* (Horoscopy of the Greeks) of *Sphujidhvaja* (270 CE), who, according to his claim in the text, versified an earlier prose version of *Yavaneśvara* (150 CE), a great exponent of Greek astrology. However, this example is cited by Kaplan[34] in favour of his argument that '*...the hollow circle of kha and the solid dot of bindu came to India from Greece*'.

[25] Also a Kashmirian version of *Atharvaveda* is known, where many examples of zero being represented by dot, both in the marginal notes and in the text itself, are found. While the numbers in the marginal note may be accounted for as later inclusion, "*the symbols in the text itself must be from the time of* Atharvaveda *(500 BCE)*", observes Mukherjee[36], a minority view, of course.

[26] *chidraṃ kham ity utkam ; Gopatha Brahman* 2.2.5[44]

[27] It is preserved in the Bodleian Library at Oxford University.

[28] However, one scholar interprets one particular dot among them to be actually a small circle[36]

[29] The colophon to the section on *trairāśika*, 'the rule of three' reads: *This has been written by the son of Chajaka, a brāhmaṇa and king of mathematicians, for the sake of Hasika, son of Vaśiṣṭha, in order that it*

the date of this work range from as early as the 2nd Century of BCE up to the 12th century CE, however as per some recent writings it is believed to be not much later than 7th century CE.[30]

As far as the first written undisputed[31] evidence (stone inscription) of the use of 'zero' as a decimal digit is concerned, depicted as a neat round circle and found in Indian soil, as of now, one may find it in the two stone inscriptions of Gwalior[32] (875/876 CE), discovered[33] in the nineteenth century.

S.C. Kak[31],[32],[33] while working on the origin of the sign of zero in India, explored possible links between Brāhmi numerals and some Indus symbols. He came to the conclusion that '*the Indus 5 and 10 became the corresponding Brāhmi signs*', from which he proposes to push back '*the ancestry of the zero sign to the third millennium BCE.*' However, since no decipherment of Indus scripts are yet accepted in general, this can be regarded as a minority view.

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may be used (also) by his descendants.[23]

[30]G.R.Kaye put it in 12th century CE,(however his view is now rated by many scholars as biased, Joseph[28] has critised him in very strong words : *it is particularly unfortunate that Kaye is still quoted as an authority on Indian mathematics*) Hoernle[24] assigned 3rd to 4th century CE, B.B. Datta[11] gingerly at early centuries of CE, L. Gurjar places it in between 2nd century BCE to 2nd century CE, A.A.K. Ayyangar in 8th to 9th century CE, I. Pearce[47] in sometime before 450 CE, (with a hint that the current version is a copy of an earlier work), whereas R. C. Gupta[20], Kaplan[34] and Hayashi[23] in 7th century CE.

[31]The inscription of Gwalior referred to here is by no means the oldest inscription, if one includes the charters inscribed on copper plates as well. At the beginning of 20th century CE, some scholars, mainly G.R. Kaye (who is by now well known for his Eurocentric bias) and some others who share his views, questioned the authenticity of these plates, claiming them to be possible later forgeries, without much of concrete evidence in favour of such a claim. Towards trying to establish the Greek priority as the real inventor of our numeral system, their prime objective was to establish that the place value system with a clear use of zero in it, could not have been there in India much before the second half of 9th century CE. However their viewpoint have been scrutinized closely and categorically rejected on the whole by many scholars, historians and Indologists[47],[28]. We only quote Ifrah[25] : .. *thus we can see that these authors had worked out their conclusions far better than their arguments.* If we focus our attention on the inscription of the numeral 'zero' only, there are several unquestionably authentic documents (stone inscriptions) found in various places of Southeast Asia, places which came under the cultural dominance of India at that time. The earliest known such epigraphic documents date back to 683 CE, one being the Khmer inscription of Trapeang depicting the numeral 605 and the other one is the Malay inscription at Palembeng (Sumatra) depicting 606.[25]

[32]This place is about 300 km. south of Delhi, in present day India. The inscriptions are from King Bhojadeva's reign and were found in a temple of Lord Visnu.

[33]Written in Sanskrit, using *Nāgarī* numerals, the first one contains, among other numerals, the numbers 10 and 20 (as the number of corresponding stanza of the inscription), while the second one shows the numbers 270 and 187 in the context of the area of the piece of a land being offered to the God, to be converted into a garden of flowers, so as to offer 50 garlands of flower daily.

Mathematical operations with the digit *śūnya*

Apart from the *Bakṣālī* Manuscript, of which the time frame is hotly debated, many examples of two fundamental operations by the zero (referred to as *kha, śūnya, ambara*) viz. that of addition and subtraction can be found in *Pañchasiddhāntikā* of *Varāhamihira* (505 CE). However, these are described in language and not by symbols[10]. Clear mathematical computations involving zero along with the formulations of rules necessary for such calculations[34] are found in *Brahmagupta*'s[35] (b. 598 CE) work *Brāhma Sphuṭa Siddhānta* (completed in 628 CE). He treated 'zero' (referred to as *kha*) as a separate number, neither positive (*dhana*) nor negative (*ṛṇa*). In chapter seven, on *Gaṇita*, while dealing with the formulations of norms of operation among various types of numbers, he states in clear terms

'*dhanayor dhanam ṛṇam ṛṇayor dhanarṇayor antaraṃ samaikya khaṃ*',

which means, '[the sum of two] positive [number]s [is] positive, [that of two] negative [number]s [is] negative, [the sum of one] positive [number and one] negative [is their] difference, [the sum of] two equals [opposite in sign, is] zero'[60]

— a mathematical principle to the effect that $x - x = 0$, treating zero as a number in its own right. From his time onwards, one may find in the Sanskrit texts on mathematics (mainly devoted to Astronomy) a separate section entitled *śūnya-gaṇita* , i.e., computation with zero, which clearly indicates that the special status of zero in mathematical calculation was strongly established in the Indian academia by that time. Many more examples intermingled with the concept of place-value arithmetic is given in the next section.

6 Decimal Place Value and Zero

"India gave (the) world a priceless gift—the decimal notation. This profound anonymous innovation is unsurpassed for sheer brilliance of abstract thought and utility as a practical invention. The decimal notation derives its power from two key strokes of genius : the concept of 'place-value', and the notion of zero as a digit."[16]

Undoubtedly, the concept of zero is most instrumental in making a place-value system work. It may apparently seem quite nonsensical an idea 'to have a symbol that represents

[34]However, his attempt to define division by zero *khacheda* ('*cipher divided by cipher is naught*') is not accurate. Modern Mathematics rules out 'division by zero'. Raju[50] opines on the contrary, a view worth mentioning. He put forward a completely different perspective of śūnya, invoking 'infinitesimals' of non-standard analysis introduced by Abhraham Robinson in 1960.

[35]He was ascribed with the appellation *Ganakacakracūdāmani*, 'crowning jewel among the circle of mathematicians' by Bhaskara II, an eminent mathematician of later period (b. 1114 CE).

nothing', but this idea, in harmony with the other numerical symbols[36] needed for the system, is precisely what makes a positional numbering i.e, a place-value based enumeration system work. Since there exists infinitely many numbers, to be able to write any one of them, however large, by merely using ten symbols (hence called the 'decimal' system, nine symbols for one to nine and the tenth one, for zero) was a truly a prodigious intellectual achievement of the Indian scholars of remote past.

It is worth quoting the famous French mathematician Pierre Simon De Laplace (1814) in this regard :

> *The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place-value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculations and placed arithmetic foremost among useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of antiquity, Archimedes and Apollonius.* [25]

The principle of place-value system assigns to each digit of a number a certain value (called the 'place-value') by virtue of its position (i.e., place) in that number. The decimal place value system, for example, assigns the place value one to the 'place' in the extreme right, often called the units place, the 'place' immediately left to it gets the value ten, followed by the 'places' of hundred, thousand, ten thousand and so on in a leftwardly direction. Clearly, each 'place' has a place-value which is a certain power of ten, hence the name *decimal*. For example, if we look at the number 2009 in this system, the digit (called *aṅka* in Sanskrit, which literally means 'mark') 2 having the intrinsic value two, stands for the value 'two thousand' by virtue of its being in the thousands place, while the digit 9, being in the units place, represents the value nine, which happens to be the intrinsic value of that digit as well. Observe that the two zeros standing in the middle, respectively occupying the tens and the hundreds place, do not apparently show up in the *number name* of 2009 (which is *two thousand nine*) but they do contribute to it, from behind the screen, in the sense that, it is their presence in the right 'place' that have allowed the digit 2 to occupy its correct 'place'

[36]For example, if zero is put at the right side of a number written in a place-value system, say binary (i.e., base 2), ternary (i.e, base 3) or, in general n-ary (i.e, base n), the original number gets multiplied by its 'base' number n. It is the lack of this characteristic feature, that compels us to ignore the Mayan claim over the inception of 'true zero' in an otherwise fascinating place-value based intricate system derived by them. Mayans lived in the isolation of Yucatan peninsula and reached their peak during 250 BCE -900 AD. They used a sea-shell like symbol, along with nine more pictorial glyphs to represent their 'zero' in different contexts[25][pp. 321]. How this civilization, with a definite passion for arithmetical calculation, almost amounting to an obsession, suddenly vanished into the thin air, is a mystery yet to be resolved.

for the decimal shaping of this number. This is an example of the powerful role of the digit zero as a placeholder. In the language of G.B. Halsted[21]:

> *The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, symbol, but helpful power, is the characteristic of the Hindu race whence it sprang....No single mathematical creation has been more potent for the general on-go of intelligence and power.*

In a recent well-versed exposition A.K. Dutta[16] aptly poitned out that, '*In the decimal representation through words, terms of powers of ten plays the role of place-value principle*'. Such concept can be found in the early Vedic text called *Vājasaneyī Saṁhitā* (XVII.2)of *Śukla Yajurveda* in a prayer of seer *Medhātithi*, where one can find the names of specific terms for each power of ten up to a billion, where each term is defined to be ten times of the preceding one (for example, *daśa daśa* a *śata*, i.e., ten times ten is a hundred, *daśa śata* a *sahasra*, i.e., ten times hundred is a thousand, etc., going up to 10^{12}, called *parārdha*)[37]. One may further refer to Gupta[18] for similar examples in other *Saṁhitā* and *Brāhmaṇa* texts. Instances of number names in conformity with decimal place-value representation can also be found in *Ṛgveda*— the oldest text among the Vedic corpus. Take for example, *sapta śatāni viṁśatiḥ* i.e., *seven hundred twenty* [I.164.11], and *sahasrāṇi śatā daśa* i.e., *one thousand one hundred ten* [II.1.8].[16][38] Ample evidences can be cited to claim that right in the era of the *Saṁhitā*s, any number at least up to a billion, if not more, could be represented by word numerals verbally, using the nine word numerals (viz. *eka* (one), *dvi* (two), *tri* (three), *catur* (four), *pañca* (five), *ṣaṭ* (six), *sapta* (seven), *aṣṭa* (eight),*nava* (nine)) and the names indicating their position within a number in appropriate powers of ten. Dutta[16] observes '*to arrive at a written decimal notation from the above terminology, one has to simply suppress the place-names from a given numerical expression provided one has an additional tenth numeral as a placeholder to indicate the possible absence of the nine numerals in certain places. Thus one can conclude that the structure of the Sanskrit numeral system contains the key to the decimal place value system.*'

[37]For the actual verse along with its translation in English, see Dutta[16] See also pp. 30 of Sarma[53] for a similar example in *Śivasaṅkalpopaniṣat*

[38]However S.A. Paramahans[46] cites a counter example from *Ṛgveda*. The hymn 3.9.9 reads as *trīṇi śatāni trī sahasrāṇy agnim trimśac ca devā nava cāsaparyan*, which refers to the number 3339 being expressed as *three hundred three thousand and thirty and nine*, apparently not in coherence with the place-value notation and more akin to additive principle. This motivates him to conclude, '*...ancient Indians introduced the decimal place-value after the early Vedic period....*'. But in presence of so many examples in tune with decimal place-value system found in various early Vedic texts, only a tiny fraction of which is cited in this article, we would rather take this one as a stray incident, occurred perhaps due to some metrical compulsion of composition.

There are evidences to suggest that Buddhist scholars of ancient time were also well versed in the use of decimal place-value system and that knowledge of *Gaṇita* was considered important. The period from 500 BCE to 900 CE coincides with the rise and dominance of Buddhism. In the *Lalitavistāra*, biography of *Gautama Buddha (b. 562 BCE)*, which may have been written around the first century CE, one finds *Buddha* enumerating to a mathematician *Arjuna*, upon being challenged by him, successive number names in powers of 10, beginning from a *koti* i.e., 10^7 up to 10^{53}, *tallakṣaṇa*.[39]

In *Vālmīki Rāmāyana*, while giving the exact strength of Rāma's army, one may find (6.28.33-41) the use of *lakṣa* scale, in which the next term is 10^5 times of the previous one, giving terminologies of numbers up to 10^{55} (*mahaugha*) beginning from 10^7 (*koṭi*). A.K. Dutta has cited plenty of references[16] of place-value system being used in the various early Sanskrit texts, enumerating numbers up to as large as 10^{421}. Among them, one example of reference to decimal system is found to be in the great Hindu epic *Mahābhārata*. In the *Vana Parva* section, one finds a scholar, *Bandi* by name, while debating with *Aṣṭāvakra* in the court of king *Janaka* saying :*navaiva yogo gaṇaneti śaśvat*, which means *nine is the perpetual number of symbols one combines in calculation*. B.B. Dutta (who later became known as *Swāmi Vidyāranya*), a pioneer among the historians of mathematics in early India, observed that, if the names involved in the aforesaid instance truly reflects the antiquity of the legend, then it will not be far-fetched to infer that the decimal place-value system was in use during the *Brāhmana* period of the Vedic culture.[51] However, this does not form a strong evidence, as the exact time of original composition of this epic is uncertain and it is believed to have gone through several later alterations and additions.

The sufficiency of nine symbols, as mentioned above, if taken verbatim, may raise some eyebrows. Indeed, one has to presume that there was another symbol, denoting the absence of those nine, as otherwise, without such a notation, nine symbols would not have been adequate. However, time and again, a school of scholars, sometimes referred to as the 'Euro-

[39] Kaplan[34] argued at length to establish that this is a direct example of Greek influence, referring to the famous problem known as '*sand-reckoner*' by Archimedes (born c.287 BCE). However one must observe that, while the sequence of numbers occurring in the story of Buddha is purely decuple in nature, what Archimedes cooked up as numbers of high order in sand-reckoner was based on clumsy Greek system of enumeration, in which the largest name of a number was 'myriad', merely 10^4. Of course, it was sheer genius of Archimedes that made him declare a myriad myriad as a number of first order and then taking it as unit and repeating the similar process phase after phase, he proceeded further to higher and higher orders of numbers. But this does show, conceptually how very far away from a place-value system, let alone decimal, he was working! Kaplan highlighted the apparently proud exaltation of Buddha, uttering in the moment of his ultimate glory "beyond it is the incalculable" to declare, '*there is no notion yet of a full positional system (if there were, we would see that enumeration cannot end........)*'[34][pp. 40]

centric' school, (as according to their doctrine, whole of the ancient mathematics is believed to have their European roots, be it in Greece or in Babylonia) has pointed out to similar instances and examples[40], particularly pointing out to the early Latin works/translations of the Arabic assimilations of Hindu works on mathematics. Some recent arguments in this direction may be found in Kaplan[34][pp. 46].

A clear formulation of a rule pertaining to the decimal place-value based enumeration is found in the works of the famous *Āryabhaṭṭa* I (b. 476 CE) of *Kusumapura*[41] who writes

sthānāt sthānaṃ daśaguṇaṃ syāt [42],
which can be translated as,
(in a number) from one 'place' to the next, the place value is increased ten times .

Jinabhadra Gani (529-589) in *Bṛhat Ksetra-samāsa*, depicting a clear knowledge of zero in a place value system, gave the number 224400000000 as

dvi vimsati ca catur catvāriṃshati ca aṣṭa śhūnyāni
"twenty-two and forty four and eight zeros"[25], [11]

Another example found in B.B. Datta, from the middle or end of the sixth century is of *Siddhasena Gaṇi*. While commenting on *Tattvārthādhigama-sūtra* of *Umāsvāti*, a Jaina[43] mathematician of the *Śvetāmbara* sect, *Siddhasena* employed decimal place value numerals in his calculations, as has been elaborately shown in B.B. Datta[11]. Again, in the commentary by *Vyāsa* on *Patañjali*'s *Yogasūtra* (3.13) (7th century CE) one finds the following principle of enumeration:

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[40]In the words of Severus Sebokht, a Syrian monk(662 CE) :*I shall not now speak of the knowledge of the Hindus,....of their methods of calculation which no words can praise strongly enough—I mean the system using the nine symbols.*[4]
In the words of a Spanish Monk, Vigilán,(976 CE), found in a writing referred to as the Codex Vigilanus, now kept in the library of the monastery of El Escorial, Spain[15] : *The Indians have an extremely subtle intelligence....The best proof of this is the nine symbols with which they represent each number no matter how large.*

[41]This place is believed to be near Patna of Bihar in modern India.

[42]*Āryabhaṭīya*, Ganitapada, verse 2

[43]Jainism arose as a revolt against the Vedic practice of sacrifice and is perhaps as old as the Vedas, if not older. Once it was thought that a contemporary of *Gautam Buddha*, *Vardhamāna Mahāvīra* by name, was the founder of jainism. But now historians are certain that he was only the last *Tīrthankara*. An interesting aspect of Jaina philosophical concept of zero as a number and a non-number is found in *Śrī Bhuvalay* of *Kumudendu*, in a narration of an ancient legend relating to the first *Tīrthankara*, *Ṛṣabhadeva.*[57] Jainas had a rich tradition of mathematics, mostly known from the later commentaries, many an original text being not extant.[61]

yathai'kā rekhā śatasthāne śataṃ daśasthāne daśai'kā cai'kasthāne....[42]
just as the rekhā *(denoting one) is called a hundred in the 'hundreds' place,*
ten in the 'tens' place and one in the 'units' place......."

In a recent encyclopedic work on this subject, Ifrah,[25][pp. 416-418, quoting a personal communication of Billard on an anonymous work published from Sholapur in 1962] argues at length to present an irrefutable proof of existence of decimal place-value system with zero being used in harmony with it, in a 5th century CE Jaina work on cosmology, entitled *Lokavibhāga* (meaning, *The Parts of the Universe*). Interestingly, using astronomical information available in two particular verses (no. 52 and 53) of the text itself, one can decipher the exact date of its completion to be precisely on *Monday, 25 August, 458 CE*, whereby Ifrah concluded it to be the '*the oldest known authentic Indian document to contain the use of zero and decimal numeration.*' However this is not the end of the story. Translating the content of the verse no. 51, Ifrah points out that '*These could and very probably does mean that the current version of the* Lokavibhāga *is an exact (*Sanskritized*) reproduction of an original (*probably written in a Jaina dialect of Prākrit*) which was written before 458 CE*, by some *Muni Sarvanandin*. But when did this *Muni* live? A hundred or few hundred years back? With a touch of frustration, Ifrah comments '*We will never know.*' Perhaps posterity will!

Without making the list any longer[44] we conclude this section by quoting the praising remarks of A.L. Basham[4] towards the unknown inventor of the decimal place-value system:

> "*Most of the great discoveries and inventions of which Europe is so proud..... would have been impossible if Europe had been shackled by the unwieldy system of Roman numerals. The unknown man who devised the new system was from the world's point of view, after the Buddha, the most important son of India. His achievement, though easily taken for granted, was the work of an analytic mind of first order.* "

[44] For an almost unending array of examples one may refer to Ifrah.[25]

7 Gradual Genesis of 'ZERO' from *Śūnya*

The Sanskrit word "*śūnya*" was transliterated into Arabic in Al-Khwarizmi's[45] (780-850 CE) work *De Numero Indorum*[46] as '*sifr*', meaning '*empty*'. The associated Arabic root $\sqrt{sfr}$ leads to other words of similar meaning, like *asfara* (meaning 'empty') or *safir* (meaning 'to be empty'). The slightly changed term '*sifra*' was used to denote 'zero' in *Sefer ha misper* (meaning, 'Book of Numbers') by Rabbi Ben Ezra (1092-1167). This one along with many local variants like *cifra, cyfra, cyphra, zyphra* etc. were used for the same purpose, i.e., to denote 'zero'. However, they took a turn by the beginning of the Renaissance and one may find the Spanish word '*cifra*', derived directly from Arabic '*sifr*' being used with a changed connotation to refer to the rest of the digit numbers. Similar examples prevail in case of other European languages as well—for instance, '*chiffre*' in French, or '*ziffer*' in German are still used to refer to digit numbers. However, the Arabic *sifr* was gradually latinized as '*zephyrus*',—'the west wind' and Leonardo of Pisa, better known as Fibonacci, while writing his seminal book, *Liber Abaci* in 1202, coined the term '*zephirum*'— mere light breeze— almost nothing, to denote 'zero'. Later it got converted to *zefiro* in the Venetian dialect of early Renaissance, and our 'zero' is a mere contraction of that[47].

8 Early Non-Mathematical Roots of *Śūnya*

Since written mathematical treatise of remote Indian antiquity is not extant, historians have turned their attentions to the non-mathematical texts of that period, with an intention to find tacit mathematical clues, if there be any. The idea is to dig deep into the academic ambience of that period and analyze in the light of those works, the available social contexts, towards possible unearthing of some threads of mathematical thought, which might have been prevalent during that period. If non-mathematicians are found to be at ease with novel mathematical concepts, for example, that of 'zero' as a digit in its own right, in an arguably

[45]Often referred to as the Father of Algebra, he worked in the *House of wisdom*(*Bait Al-Hikma*), an institution created in Baghdad for the advancement of philosophy, astronomy and mathematics, during the rule of Al-Ma'mun, the caliph who is mentioned in the famous *One thousand and one Arabian Nights*. His treatise on Arithmetic, called *De Numero Indorum*, is known to us through a XIII century version in Latin, as its Arabic version has not survived.

[46]Around 750 CE in the court of Caliph Al-Mansoor, a copy of an Indian mathematical work, which some historians argue to be *Brāhma Sphuṭa Siddhānta* by *Brahmagupta*, was presented and later translated by the name *Sindhind* and presumably this motivated the arithmetical work of Al-Khwarizmi.

[47]The first known occurance of the modern term 'zero' is found in an Italian book *De arithmetica opusculum* by Philippi Calandri, pulished in 1491 in Florence.

place value-system[48], it certainly strengthens the argument in favour of those peoples doing *Gaṇita* in that society to have mastered those concepts. Two such cases are certainly worth mentioning, one directly referring to the 'zero' of mathematics and the other tacitly applying a similar concept.

8.1 Direct Evidence : *Śūnya* in *Piṅgalacchandaḥsūtra*

The *Piṅgalacchandaḥsūtra* i.e., the rules of prosody for both Vedic and Classical Chandas given by *Piṅgala*[49] [also known as or referred to as *Piṅgalācārya*,(in Halāyudha) *Piṅgalmuni* (in *Mahābhārata*, where he is found to be the chief priest in the serpent sacrifice of King Janmejaya to take revenge of his father's death[50]) or *Piṅgalanāga* (in Jivānanda Vidyāsāgara[62])] comes under the limelight as an irrefutable landmark in the history of mathematics, particularly in connection with the priority in the inception of place-value notation, as in two of its *śutras* it formally mentions the term '*śūnya*' for the first time in the history of human civilization (as far as our present evidences stand), a clear mathematical concept of 'zero' and discussed formulations for intricate combinatorial calculations towards what is now recognised as binary arithmetic, going up to conversion of a decimal number to binary and vice versa— an extraordinary feat of achievement indeed.[42]

Analyzing the reasons behind abundance of metrical verse based resources of Indian Antiquity and the consequent indispensability that arose towards careful and systematic analysis of poetic metrics, A.K. Dutta concludes '*Ancient Indians had the perception that the metrical form has greater durability, power, intensity and force than the unmetrical, and hence recorded all important knowledge in verse form. In particular, all ancient Indian mathematical literature, beginning with the pre-Pāṇinian* Śulba-sūtras*, are composed entirely in verses.* This tradition of composing terse *śutras*,[51] (*often by avoiding the use of verbs as far as possible and compounding nouns at great length*) which could easily be memorized, ensured that, some of the core knowledge got orally transmitted to successive

[48] *Piṅgala* employs *bhūtasaṃkhyā*(i.e., word numerals) rather frequently. But in all these places he uses the words singly. Therefore, his employment of word numerals does not clearly indicate place-value.

[49] The word comes from the root $\sqrt{pigi}$, derived as *piṅgayati iti piṅgala*, meaning *that which gives light*[56]

[50] It is not settled beyond doubt whether this priest is the same person as the prosodist under discussion. Indeed, going perhaps a bit too far, Van Nooten observed "we do not know who the author Pingala was, we do not know where he lived, when his work was composed and finally, whether the work going by his name was really all his, or a product of his school, or a conglomerate of texts fragments assembled at one time and then forth transmitted under his name." However, though the first two doubts are shared by many scholars, the final one is not corroborated in their findings.

[51] In *Viṣṇudharmottara purāṇa*, one finds an apt definition, translated by Sridharan[61] as "A *sutra* should have the least number of syllables, should contain no doubtful words, no redundancy of words, should have unrestricted validity, should contain no meaningless words and should be faultless"

generations.'[16]

The Vedic seers attributed almost mystic significance to *chandas*,[52] the fifth *Vedāṅga* dealing with the meters of the sacrificial chants. Consequently, close attention was paid to the study of *chanaḥśāstra* i.e., the science of verse meters, along with its language, prosody and the proper time and place for its recitation. It is stated in the *Śruti* that, he who made rituals possible by the brahmins, should know the meter of each *mantra*. Before the study of a *mantra*, one must have a clear knowledge about the deity, meter and the seer associated with it, failing which the rituals performed will not attain the desired goal.[56] Further in this regard, one may also quote from *Bṛhaddevatā*, [verse 136, VIII] supposedly written by the Vedic seer Śaunaka:

aviditvā ṛṣiṃ chando daivataṁ yogameva ca
yo'dhyāpayejjapedvāpi pāpīyāñjāyate tu saḥ
One who teaches or recites the Veda without having proper knowledge of the application, the seers, meters and Gods, becomes indeed a sinner.[61]

Hence it is no wonder that the study of analyzing the meters became a very respectable study right from the early Vedic society onwards. *Piṅgalacchandaḥsūtra* is the only authentic extant complete work, though not the earliest, in this field[53]. The unknown author of *Agnipurāṇa* refers and summarises the work of *Piṅgala*. He states:

chando vakṣye mūlajaistaiḥ
piṅgaloktaṃ yathākramam
I shall now state about the chandas
in the order uttered by Piṅgala, our Elder

There exist many commentaries on it, done at much later dates, the most referred ones include *Kedāra Bhaṭṭa*'s *Vṛttaratnākara* (1150 CE), *Yādav Prakāśh*'s commentary (circa 1050 CE) and*Mṛtasañjīvinī* by *Halāyudha*(11th century). Though there were many writers of

[52]There are two kinds of derivations of the word *chandas*, '*laukika*' (earthly) and '*alaukika*' (unearthly). In the *Nirukta* of *Yāska* one finds '*chandāṁsi chādanāt iti*' i.e., 'chandas' comes from the root $\sqrt{}$*chad*, which means 'to cover'; in Pāṇini grammar, the word comes from the root $\sqrt{}$*cand*, derived as *candayati hlādayati iti chandaḥ cander ādeśaś ca chaḥ*.[56](pp.51) But what were they suppose to 'cover'? In the *Aitareya Āraṇyaka, adhyāya I, khaṇda V, āraṇyaka II*, one may find in the context of worshiping the *prāṇa* God, that He is to be meditated as *covered* by the meters, whence the meters are *chandas*. It states further, *chādayanti ha vā enaṃ chandāṁsi pāpāt karmaṇo yasyāṁ cid api kāmayate ya evam etac chandasāṁ chandastvaṁ veda iti*, which means, *he who knows meters as the cover of the* prāṇa *God will be saved from his sins*[56](pp.52).

[53]The German Indologist Albrecht Weber was the first to make a critical study of the *Chandaḥsūtra* as early as 1863, under the title "*Über die Metrik der Inder*" in the eighth volume of the journal *Indische Studien*.

metrical books earlier to *Piṅgala*, as is testified in his book itself, unfortunately only the names of their authors are available, their works have presumably perished with time.

Now what attracted the attention of the historians of mathematics, particularly those dealing with antiquity, towards the work of this master prosodist, *Piṅgala*? Indeed, after a critical examination of all the 315 *sūtras* spread over the eight chapters of his work, one finds the *sūtras* 20-35 in the eighth chapter to be of immense mathematical potential. As usual, they are in keeping with the spirit of pithy and cryptic presentation of a '*sūtra*', almost obscure to a non-specialist. However, thanks to the later commentaries, that they can be relatively easily deciphered to produce an excellent account of combinatorial calculation along with binary arithmetic.[42][61] We narrow our focus of attention to the *sūtras* 28-31, dealing with the combinatorial question, how many different meters are there with a given length? That is, to compute the total number of possible arrangements of long and short syllables (respectively, the *guru* and the *laghu*), repetition allowed, without actually constructing the arrangement of all possible combinations of *guru* and *laghu* in a given meter (called the *prastāra*, a bed or matrix in which the *guru*s and *laghu*s are listed horizontally)[54]. Of our particular interest in this article are the *sūtras* 29-30, *rūpe śūnyam* and *dviḥ sūnye* by name, clearly mentioning '*śūnya*'. However to see them work as mathematical '*śūnya*', one has to combine them along with the *sūtras* immediately preceding and immediately succeeding them, viz. *dvirardhe* and *tāvad ardhe tad guṇitam*. Now four of them, put together in a row, and interpreted according to the commentators, give us the necessary rule of computation, that clearly involves a notion of mathematical "zero". The four sutras are beautifully translated in English by Sharma[55] as follows[55]:

sūtra **28** : "[First write down the number of syllables in the given meter and go on halving that number. Each time] **when** [the number is] **halved** (*ardhe*), [write down in a separate row or column the digit] **2** (*dviḥ*).

sūtra **29** : "[When you reach an odd number, subtract 1 from it.] **Whenever 1**[56] [is subtracted (*rūpe*), write down in a separate column a] **zero** (*śūnyam*).

sūtra **30** : "[Continue thus until the process stops. Then where you wrote a] **zero**(*śūnye*), [multiply by] **2** (*dviḥ*).

sūtra **31** : "**Where** [the number was] **halved** (*tāvad ardhe*), **multiply** [the result of the second process] **by itself** (*tad guṇitam*)."

[54]While dealing with these constructions, *Piṅgala* considered only those meters where all the four feet of the verse have identical pattern, called *samavṛttas*. A detailed lucid account of the actual calculations may be found in [42], [61], [55]

[55]The bold letters indicate actual word by word translation, whereas the completion of the sentences are done according to the commentary of *Halāyudha*. This is a good example to cite how terse the presentations of *sūtras* used to be.

[56]*rupa*, meaning 'moon' used to stand for 'one'.

Following the above rule we may construct the following concrete example, pertaining to the *uṣṇik* meter having total twentyeight syllables, whence each *pāda* in its *samavṛtta ctuṣpāda* variation has seven syllables. We are to find the number of all possible arrangements when each one of the seven places may be occupied by either a *guru* or a *laghu*. Elementary knowledge of combinatorics tells us that the answer is 2^7. Let us verify it following *Piṅgala* in the table below. Columns under A and B refers to the *sūtra*s 28 and 29, whereas the column under C shows the calculation done from bottom towards top according to the *sūtra*s 30 and 31. Note that, while beginning the calculation in column C, *one has to start taking 1 by default*, and then apply the rules 30 and 31 as the case may be, a point worth mentioning, as it is not clearly stated in the (translation of) *sūtra* 30.

	uṣṇik (*samavṛtta; ctuṣpāda*)	A	B	C
1.	Write the number of syllables	7		
2.	7 cannot be halved; so reduce it by 1	6	0	$(2^2 \cdot 2)^2 \cdot 2 = 2^7$
3.	Halve 6	3	2	$(2^2 \cdot 2)^2$
4.	3 cannot be halved; so reduce it by 1	2	0	$2^2 \cdot 2$
5.	Halve 2	1	2	2^2
6.	1 cannot be halved; so reduce it by 1	0	0	$1 \cdot 2$
		Stop.		Begin upwards

So it appears that *Piṅgala* prescribed to use two symbols, viz. that of two and zero,[57] towards distinguishing between two kinds of operations, squaring and multiplying by 2. Surely, such a calculation could have been done with any two arbitrary symbols. But, his choice favouring the term '*śūnyam*' clearly indicates that during his time a concept of mathematical *śūnya* was prevalent. But what kind of *śūnya* was it? Did it only signify absence of an operation, in a spirit similar to the *Pāṇinian* concept of *lopa*, or that of a decimal place-value system in vogue? Van Nooten[42] observes that, "*we can be reasonably certain that his counting system was predicated on a base of ten.*" He further points out that a system of notation that simply uses two symbols as markers of a place value does not necessarily produce a binary system. But if there is a system that uses two symbols in such a way that every string made with them has a unique decimal equivalent and it is shown that how one may convert a decimal number into a string of those two symbols uniquely, "*then indeed we do have a binary system.* Piṅgala *has done at least this much.*" In the view of Needham[40], "*Place value could and did exist without any symbol for zero. ...But zero symbol as a part of the numerical system never existed and could not have come*

[57]However, his treatise does not provide us with any representation of numerals. Wherever he had referred to *śūnya*, he did it in language and not by any symbol. So whatever symbol for *śūnya* he might have had in his mind, remains beyond us.

into being without place-value." Along this line of argument Sharma[55] concludes "*the invention of decimal place-value system along with the concept and symbol of zero must antedate considerably* Piṅgala*'s mention of the zero symbol.*"

The time of the original composition of *Piṅgalacchandaḥsūtra* is as usual hotly debated among the scholars. But setting his time frame accurately, as far as possible from the circumstantial evidences, is extremely crucial, as this may pave the path towards a rejection of the so-called Eurocentric view of Greek origin of the concept of zero and its being transmitted to the Indian soil through Alexandrian invasion of the northern india (326 BCE)[59](pp.63)[34](pp. 37). In the resulting tug-of-war we find Sastri[56](pp.81) placing him in 4th century BCE while Dutta and Singh[9] and Dowson[14] argue to place him around at least 200 BCE, if not earlier. Again, S.R. Sharma[54] is in favour of keeping the time-frame a bit wider, between 400 to 200 BCE, which is contested by Bronkhorst[7], who not only claims that the period will be much later but also suggests that the eighth chapter of the book, where the clear mention of *Śūnya* can be found, is a later work, not original to *Piṅgala*. On the other hand, Van Nooten[42] in his paper, while attributing the credit of invention of binary numbers to *Piṅgala*, recognizes the eight chapter to be original[58], but analyzes the time-frame critically and is not ready to put him before 3rd Century of CE. In his opinion, since the earliest known reference of *Piṅgala* is found in *Mīmāṅsāsutra* of *Śabara*, whose time has been assigned to 4th century CE, and as 'the internal evidence of the treatise does not militate against a date before the 2nd century A.D. since the composition of the *sūtras* follows the pattern of the older universified *sūtras*, such as those contained in *Pāṇini*'s *Aṣṭādhyāyī*', according to him 'it is not possible, on objective grounds, to decide whether *Piṅgala*'s treatise preceded or followed *Pāṇini*, nor is it possible to prove that *Piṅgala*'s work existed before the third century A.D.' But Sharma in a comparatively recent article[55] strongly refutes this claim and put forward fresh arguments and views to the effect that the eighth chapter is indeed original to *Piṅgala*[59] and "there are close similarities in the methods of exposition and notation between the *Chandaḥsūtra* on the one hand and the *Aṣṭādhyāyī* and the *Vedāṅgajyotiṣa* on the other. The affinity with these two texts places the *Chandaḥsūtra* at about 400 BCE."

All these show the importance of a comparative time-frame based study of cross-references

[58] The passage where the binary system developed is in all likelihood part of the original work[42](pp.32)

[59] Sharma observes that[55], '*The* Pratyaya *section of the eighth chapter is not a loose appendage but is anticipated in the preceding chapters of* Chandaḥsūtra.*' Indeed, the treatise begins with the definition of the eight* gaṇas *or triplets made of all possible permutations with two kinds of syllables,* laghus *taken as* l *and* gurus *taken as* g*. They are then given names as* m*, the* ma-gaṇa *consisting of the arrangement ggg; then comes* y*, the* ya-gaṇa *consisting of the arrangement* lgg*; next is* r*, the* ra-gaṇa *consisting of the arrangement* glg*; etc. Sharma[55] claims that, 'this sequence of triplets make sense only in the context of the* Prastāra *of the eighth chapter........otherwise,* Piṅgala *could have enumerated the triplets in any other sequence'*

available on the two literary giants of Indian Antiquity, *Pāṇini* and *Piṅgala*. One can find a point by point comparison in A. Sastri[56]:

Pāṇini dealt with *sūtras* both *laukika* and *vaidika*; *Piṅgala* dealt with *chandas* both classical and Vedic.

Pāṇini took recourse to *pratyāhāras* to make his *sūtras* concise; *Piṅgala* reflects a similar spirit introducing the *gaṇas* in analyzing *chandas*.

Pāṇini begins with *vṛddhirādaic* as an auspicious *sūtra*; *Piṅgala* took a similar stance with the *dhīśrīstrīm* which suggests auspiciousness.

Pāṇini at first has given the definition of lengthening; *Piṅgala* the lengthened syllables.

Pāṇini used the word *vṛddhi* as *nāndī*; *Piṅgala* in turn used *dhī*, a term depicting practically the same connotation.

In the first verse of *Pāṇinīya Śikṣā* one has

atha śikṣāṁ pravakṣyāmi
pāṇinīyaṃ mataṃ yathā
I now talk about phonetics, according to the doctrine of Panini .

The commentator of this verse interestingly remarks[56](pp. 76)

vyākhyāya piṅgalācārya-
sūtraṇy ādau yathā yathā
śikṣāṃ tadīyāṃ vyākhyāsye
pāṇīnīyānusāriṇīṃ

Upon having explained the rules of Piṅgalācārya as it is,
I shall now explain his another work śikṣā following Pāṇini.

The *Śikṣāprakāśa* clearly states that, *Piṅgalācārya*, the younger brother carefully followed the direction of *Pāṇini*, his elder brother[60], who had probably initiated him into the study of metrical science.[56] Also *Ṣadguruśiṣya* in his commentary (1187 CE) on *Ṛganukramaṇī* refers to *Piṅgala* as a *pāṇinīyānuja*[61](pp.40), the *younger brother of Pāṇini*. Interpretating 'younger brother' in the sense of being junior in the same intellectual lineage, one can now claim to have enough reasons to consider *Piṅgala* as a younger contemporary of Pāṇini. But when did Panini live? K.V. Sarma[52] sets for 600 BCE. Pandit[44] in his groundbreaking work has put him somewhere between 700-400 BCE. This leads us to conclude that, most probably, *Piṅgala* lived around third century BCE.

[60] *jyeṣṭhabhrātṛbhir vihite vyākaraṇe anujas tadbhagavān piṅgalācāryas tan matam anubhāvya śikṣāṃ vaktuṃ pratijānīte*

8.2 Indirect Evidence : Tacit concept of 'zero' in *Pāṇini*

Pāṅini, the great grammarian, the author of *Aṣṭādhyāyī*, an archetypal work, thoroughly systematizing the Sanskrit grammar, which is studied by the linguists even today as the 'work of a linguist and not a language-teacher', exhibited extraordinary technical and descriptive skills. Irrespective of the controversy on whether there existed earlier grammatical schools in Sanskrit, or how many grammarians actually predated *Pāṅini*, there seem to be no doubt among the scholars that this peerless work stands alone, 'a literary monument', among all the pre- or post-Pāṅinian grammars on Sanskrit, known to us. Out of a plethora of commentaries available on his work, two particularly important are by *Kātyāyana* and *Patañjali*, who regarded this work as a 'scientific system based on certain scientific principles' and tried to establish it through their original contributions towards the proper interpretation and understanding of the original work. Passion for attaining perfection in grammar, recognized as a *Vedāṅga* as we have already pointed out, was of paramount importance in the Vedic studies in ancient India, to such an extent that, some scholars believe, it can be thought of as an equivalent to the spirit of indispensiblity of logical rigour found in the Greek civilization of Antiquity. And it had infiltrated in their viewpoints adopted towards mathematics as well. Plofker[48] suggests that '*Sanskrit tradition does not regard mathematical knowledge as providing a unique standard of epistemic certainty. So unlike many Greek philosophers and their Islamic successors, for whom the validity of mathematical knowledge had profound implications, the corresponding role in Indian thought was filled by grammar, i.e., (*vyākaraṇa*). The previleged position of orality may have inspired the fascination with, and advanced development of phonetics and grammar among Indian Scholars. In Sanskrit philosophy and logic, ideas about reasoning and reality are explicitly linked to the understanding of linguistic statements. What philosophers need to probe in such statements, therefore, is their grammatical interpretations rather than their analogies with mathematical entities*'.

It is from this background that researches, deep into the analysis of the Pāṅinian grammar, adept to a scientific 'structure', has been carried out by scholars like Allen[1], Hass[22], Bloomfield[6] and S. Al George[17] during the middle of 20th Century CE. In the language of Bloomfield[6], '*The Hindus hit upon the apparently artificial but in practice eminently serviceable device of speaking of a zero element.*' In a later elaborate work of truly seminal nature, Pandit[44] comments '*The technique of zero is thus basically and purely a technical device born out of necessity for brevity and symmetry of linguistic description........the only technical principle that Pāṇini might have and has actually used is the principle of zero, perhaps borrowed from positional mathematics.*' Of course, there is no way one can be more categorical about this possible corelation. Indeed, A.K. Dutta[16] comments '*One does not know whether the mathematical zero existed by his time and whether he was influenced by* *it, or whether it was Pāṅini's grammar which inspired the great mathematical invention.*' But Pāṇinian techniques of linguistic analysis does consist of some basic fibers of abstract universality, making one feel that they '*are comparable to those adopted in mathematics, computers, symbolic logic and artificial intelligence.*'[20]

Pāṇini has employed three techniques named *lopa, it-samjña* and *pratyāhāra* for grammatical description and one called *anuvṛtti* pertaining to the interpretation of *sūtra*s. Pandit argued at length to establish that *all these four seem to have been based on one single principle, viz. 'lopa', which can be rendered by the modern term 'zeroing'.* These techniques by their nature are very complicated and too technical for the comprehension of general people, not well-versed with the Sanskrit grammar, and happens to be way beyond the field of specialization of the present author. However, an example in a nutshell can be found in Gupta[19]:

'*In Indo-European Philology, we have the 'zero-grade form' which is formed by dropping an 'e', for example in the following cases of roots*

$$\sqrt{\text{yeug}}\ \textit{(to join)} \longrightarrow \text{yug}\textit{; cf. yuga, yoke etc.}$$
$$\sqrt{\text{gher}}\ \textit{(to enclose)} \longrightarrow \text{ghr}\textit{; cf. gṛha, garden etc.}$$

Pāṅini's use of his grammatical or linguistic zero 'lopa' *is used as a marker of an empty (*śūnya*) non-occupied space or position. His relevant basic* sūtra *is :* adarśanaṃ lopaḥ, *found in (*Aṣṭādhyāyī, *I.I.60) which means (non-appearance [of a sound or morpheme*[61] *is lopaḥ).*[62] *That the Pāṅinian* 'lopaḥ'*, stood for a zero is also indicated by the fact that* Pūjyapāda *(c. 450 CE) in his* Jainendra Vyākaraṇa *replaced* 'lopaḥ' *in the sutra VIII.4.64 by* 'khaṃ'*, which was a common term for zero.*'

For the purpose of linguistic description of the Sanskrit language of his time, *Pāṅini* in his grammar tried to formulate a rigorous logical structure from which the whole body of the Sanskrit language might be seen to evolve. Towards this, he framed his *sūtra*s, minimum in number, each one encompassing as many cases as possible. Dealing with the verbal formation

[61] The smallest bit of language that has its own meaning, either a word or a part of a word, vide, Cambridge Advanced Learner's Dictionary.

[62] Pandit[44] however gives a much wider perspective to the term '*adarśanaṃ*' than merely 'non-appearance'. Quoting the commentary by *Kāsīkā*, he points out that, the word *adarśana* is to be interpreted as follows:

adarśanam aśravanam anuccāraṇam anupalabdhir abhāvo varṇavināśa ity anarthāntaram
etaih śabdair yo'rtho'bhidhīyate tasya lopa itīyam saṁjñā bhavati

When a sound is not heard, pronounced, available or existing or is lost anywhere, even if its place there is due on that occasion, it is said to be lupta i.e., disappeared, or in other words, 'to have amounted to zero'.

of an existing Sanskrit word, he reconstructed the evolution of it from its root according to his perception of symmetry in description and while doing this analysis, he often encountered a situation where there was neither a phoneme nor a morpheme, but to match the structure, as proposed by him, he had to account for something. *Pāṅini* proposed the concept of 'zero' (called *lopa* in his terminology) to save the situation, and one may find abundant use of this concept in his work. Following example is taken from Pandit[44]. Taking the word *daśati* from spoken language, *Pāṅini* reconstructed its root as $\sqrt{}$*daṁś*, 'to bite' and observed that its present tense, third person, singular number form should have been :

$$\textit{daṁś} + (\textit{ś})a(p) + ti = \textit{daṁśati},$$

where the nasal *ṁ* ought to be present. But it was spoken as *daśati*. Hence *Pāṅini* had to make the provision for '*lopa*' of the nasal in this case. He did it by framing the rule 6.4.25, *daṁśasañjasvañjām śapi*—i.e., the nasal of roots $\sqrt{}$*daṁś*, $\sqrt{}$*sañj* and $\sqrt{}$*svañj* disappear before *śap*, the *vikarana* (i.e., the middle morpheme) of the first conjugation, thereby making the final steps of the analysis look like :

$$\textit{da(ṁ)śati} = \textit{da(0)śati} = \textit{daśati}.$$

Thus the nasal, which was due in this case as *prasakta* (i.e., attached), was not to be pronounced, according to the Pāṅinian reconstruction.[63] Pandit submits that this use of *lopa* is much the same, structurally, with the use of 'zero' as placeholder in mathematics, while it is used in writing a numeral, say 202, to indicate the vacant place.[64]

Accepting the possibility that '*the great grammarian Pāṅini seems to have some clues for the concept of the mathematical zero*', Paramhansa[46] sceptically submits that '*the zero in* Pāṅini *has nothing to do with the mathematical zero of the place-value notation. A caution for the 'lopa' zero for Pāṅini .... It is just subjective zero as relative zero of Abstract Algebra. For example, a non-trivial normal subgroup H of a group G is the 'zero' of the quotient group G/H.*'He further gives linguistic examples of in favour of his argument. However, Pandit has categorically established with several examples[45] that the Pāṅinian technique of zero used in linguistic descriptions is comparable to the mathematical zero as a place holder, a marker for the vacant place, much like the zero in numbers 10, 20, 30 etc. To conclude this section we quote from Pandit[45] some salient similarities of the two concepts[65], 'zero in Mathematics' and '*lopa* in *Pāṅini*' :

[63] In the commentary by *Kāśika*, one may find in this regard : *prasaktasyādarśanaṁ lopasaṁjñaṁ bhavati.*

[64] In exact language of Pandit[44], '*...as in 125 and 105 in which the zero in the later indicates the place of decimal and can compare with the decimal place number 2 in the former. Though numerically zero is not equal to or identical with the number 2, from the point of view of its capacity as an indicator of the place, it is fully on par with the later. In the same way, though phonetically the, śap = 0, it is on par with non-zero counterpart so far as its place-value is concerned.*'

[65] Seven of them are given, of which we present only five, others being fairly technical.

(i) *in both of them, the concept of zero attains the status of a technical device to build up the respective systems*;

(ii) *in both of them, zero represents the zeroed entity as well as its place*;

(iii) *both presuppose positional analysis and representation*;

(iv) *in both of them, the concept of zero is not merely intellectual but can be comprehended, arrived at and represented by a definite empirical, formal process laid down in those sciences*;

(v) *in both of them, the zero requires to be inserted at the deep-structure level.*

9 Greek Connection?

As we have already pointed out, there is a school of historians of Mathematics, according to whom, the 'decimal place-value principle with zero as a hall-mark', was in actuality not an independent Indian achievement. The root of it presumably lies with the Babylonians (Akkadians), who by 300 BCE had started using a queer symbol of two slanted wedges (mostly, with certain exceptions as well[34]) to denote an empty place-mark in a written numeral on unbaked clay tablets, instead of keeping a 'blank space', as they had been doing before, for over a thousand years. Though these historians seem to give some concession to the decimal place-value system in favour of India, which according to them 'was being employed in India, especially among the Jainas and Buddhists, towards the beginning of the common era' (i.e., Christian era), as far as the inception of '*zero and its symbol*' is concerned, they opine to the contrary. Giving an undue stress towards the 'symbol' part of it, it is claimed that '*Greek (astronomical) Papyri of the period immediately preceding and following the beginning of the common era demonstrate that, they filled (the blank space) with an adaptation of the Akkadian symbol for zero (two slanted wedges); this adaptation looks like a circle with a bar over it* ($\overline{\mathrm{O}}$)[66]*. This, then, is the origin of the form of the symbol for zero*[67] *as a circle.*' Further arguments amount to the claim that, during this period, Greeks came to India in large numbers, brought with them documents with

[66]One may find the photographs of many relevant papyri in this regard in the article entitled *Consideration of the Greek symbol 'zero'* by Raymond Mercier, available in the internet.

[67]Earlier, this sign, often referred to as the *Hellenistic zero*, used by astronomers like Ptolemy (c. 140 CE) and some of his successors, only in the fractional part of a sexagesimal number (called minutes, seconds, third, fourth etc.) and not in the integral part, was thought to be emerged out of the first letter 'omicron' of the Greek word οὔδέν ('ouden'), meaning nothing. However O. Neugebaur[41] dismisses this claim by pointing out to the fact that 'omicron' was already used to mean 70 according to the Greek alphabetic enumeration system and A. Jones[27] opined in his favour. Some other scholar believe that it came from '*obol*', a contemporary coin of almost no value. They conjecture that the typical round sign evolved during the use of counters in sand-board used for arithmetical calculation. The impression left on sand when such a counter was removed to leave an empty column, perhaps gave birth to the symbol.[34]

such a symbol written there and while translating them into Sanskrit (for which the Indian mathematicians/astronomers like *Varāhamihira* are credited) 'the symbol for zero (i.e., $\overline{\mathrm{O}}$) is taken as a circle (*pūrṇa*, the full moon) or a dot (*bindu*).'[49] The earliest such incident according to D. Pingree took place around 425 CE. The principal impetus behind the whole theory being '*there is no certain evidence that a symbol for zero was in place (in India) before the 5th century A.D.*' [68] However, in the context of what we have discussed so far in the previous sections, we find such interpretations by Pingree and the co-sharer of his opinion untenable. It appears more probable that Indians even in pre-common era were familiar with the mathematical perspectives of *Śūnya*. Whatever symbol for this concept they might have used at that early stage, is of course not known to us.

10 Epilogue

Historical reconstruction of mathematical knowledge, that was likely to be prevalent in Indian Antiquity, is much like arranging an enormous zig-saw puzzle, many pieces of which are missing.[69] Competent historians of Mathematics, all over the world, are trying to arrange the available pieces, according to their own respective 'stance's, with an obvious intention to try and guess the picture it may suggest. And the job is anything but easy. Patriotic passion or pre-conceived ideas[70] often come into the way of scholarly acumen, prompting to misplace one or two pieces or perhaps not to place them at all, distorting the figure to accommodate one's 'stance'. Some are dedicated to make it look like the dancing Śiva,

[68] It however does not explain why the Greek mathematicians did not or could not use this so called 'zero' to the full potential. They were the natural inheritors of Babylonian place-value system, due to their geographical location, yet even after they were exposed to this more powerful system of enumeration, they apparently failed to judge the potential of it, used it only in astronomical calculation, that too not freely and arithmetically stuck to their own clumsy system of enumeration. Why is it that the genius of Archemedes or the galaxy of geometrical giants could not recognise it? Some believe that it was perhaps due to their overemphasis on geometric ideas, trying to interpret the universe in tandem with their perception of geometry, and this perception did not approve of the concept of 'zero'. Or was it the Aristotelian dictum— 'Nature abhors vacuum' that stood in their way, raising philosophical conundrum like 'How can nothing be represented by something?'

[69] '*India's sands were never so kind to her records as Babylonia's sands have been to her clay tablets.*'[2]

[70] One may quote G.R. Kaye (1915), as found in pp. 215 of Jacob[28] : '*The achievements of Greeks in mathematics and art form the most wonderful chapters in the history of civilization, and these achievements are the admiration of western scholars. It is therefore natural that western investigators in the history of knowledge should seek for traces of Greek influence in later manifestations of art, and mathematics in particular.*' However, Martin Bernal opines to the contrary. One may refer to his book, *Black Athena. The Afroasiatic Roots of Classical Civilization, Vol.1, The Fabrication of Ancient Greece 1785-1985,* Vintage, London, (1991)

while few others seem to be determined to find a dancing monkey! [71] Of course there is a third group, carefully steering a middle course, analysing objectively as far as possible, the available pieces of evidence or information and sometimes even the lack of information, trying to make a pattern out of it. With every new piece being found occasionally, as a new input to the zig-saw puzzle, one has to try and find its right place in the puzzle, sometimes destroying the existing pattern. And the journey continues. We put an end to our discussion by slightly altering what Bourbaki[72] once said about mathematicians :

> *Historians of Mathematics have been correcting their errors to the consequent enrichment and not impoverishment of this discipline; and this gives them the right to face the future with serenity.*

ACKNOWLEDGEMENT

The author takes this opportunity to put on record his indebtedness to the labors of the various works mentioned in the bibliography, without whose groundbreaking achievements, it would have not been possible to write this article. Thanks are due to Dr. A.K. Dutta for his illuminating suggestions and encouragement. Special thanks are also due to my family, collegues and friends for their constant support, among whom Dr. C.B. Prasad Roy, Mr. D.S. Chakraborty and Mr. P. Bhattacharya needs special mention.

References

[1] W.S. Allen;*Zero and Pāṇini*, Indian Linguistics, Vol. 16, (Nov. 1955) pp. 106-113

[2] T.A. Sarasvati Amma; *Geometry in Ancient and Medieval India*, Motilal Banarsidas, Delhi, (1999)

[3] Sri Aurobindo;*The Foundation of Indian Culture*, Sri Aurobindo Ashram,Pondicherry.

[4] A.L. Basham; *The Wonder That was India*, pp. 496.

[5] Roger Billard; *L'astronomie indienne*, Ecole Francaise d'Extreme-Orient, Paris, (1971)

[6] L. Bloomfield; *Language*, 1961, pp. 132

[71] While someone claims that '*the mathematical conception of zero....was also present in the spiritual form from 17000 years back in India*'[37], some other seems to be convinced that *Bakshālī* manuscript must belong to 12th century CE !

[72] The exact quotation is in the book, *Elements of Mathematics: Theory of Sets*, Springer 1968

[7] J. Bronkhorst; *A Note on Zero and the Numerical place value System in Ancient India*, Asiatische Studien/Études Asiatique, 48.4, 1039-42 (1994)

[8] H.N. Chakravarty; *The Concept of Śūnya in Tantra and Āgama*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.197

[9] B.B. Datta and A.N. Singh; *History of Indian Mathematics*, 2 Vols. Asian Publishing House (1935)

[10] B.B. Datta; *Early Literary Evidence of the use of the Zero in India*, Amer. Math. Monthly 33(Nov. 1926) pp. 449-454

[11] B.B. Datta; *Early Literary Evidence of the use of the Zero in India*, Amer. Math. Monthly 33(Dec. 1926) pp. 566-572

[12] R.C. Datta; *Ṛgveda Saṁhitā*, Haraf Prakasani, Calcutta, (1976) pp. 132, 292.

[13] Leo Depuydt; *Review of the Book : Unexpected Links between Egyptian and Babylonian Mathematics by J. Friberg*,The Mathematical Intelligencer, vol.30,No.3,2008; pp.72

[14] John Dowson; *A classical dictinoray of Hindu Mythology and religion*, pp.11

[15] Antonio J. Durán(edt.), *The Life of Numbers*, Biblioteca Nacional, Ministerio De Cultura, Spain, (2006)

[16] Amartya Kumar Dutta; *Mathematics in Ancient India : Some Glimpses*, *Bharatatatta*, Ramakrishna Mission Institute of Culture, Kolkata, India (2009) (to appear)

[17] S. Al George; *The Semiosis of Zero according to Pāṇini*, East And West, 17 Nos. 1-2, Rome, 1967, pp.116-119

[18] R.C.Gupta; *World's longest lists of decuple terms*, Gaṇita Bhāratī, vol. 23 (2001), pp. 83-90.

[19] R.C.Gupta; *Technology of Using Śūnya in India*,The Concept of Sunya,A.K.Bag and S.R.Sarma (edt.), Aryan Books International, pp.24 (2003)

[20] R.C.Gupta; *Some new studies and findings regarding ancient Indian mathematics*, Mathematics Teacher, vol. 38 (2002), pp. 95-107.

[21] G.B. Halsted; *On the Foundation and techniques of Arithmetic*, pp.20

[22] W. Hass; *Zero in Linguistic Description*, Studies in Linguistic Analysis, Special Volume of Philosophical Society, Oxford, 1957, PP.33-53

[23] Takao Hayashi;*Bakṣālī Manuscript*,Encyclopaedia of Classical Indian Sciences, H. Selin, R. Narasimha (edt.), Universities Press, (2007) pp. 52-55

[24] A.F.R. Hoernle; *The Bakshālī Manuscript*, The Indian Antiquity 17, (1988) pp. 33-48 and 275-279

[25] Georges Ifrah; *The Universal History of Numbers*, John Weily and sons (2000)

[26] Swami Jitatmananda; *Śūnyatā : Its meaning in Philosophy and science*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.1-18

[27] A. Jones; Astronomical Papyri from Oxyrhynchus, American Philosophical Society, (1999)

[28] G.G. Joseph; *The Enormity of Zero*, Mathematics Teacher, Ann. Conf. of AMTI, Cochin, (2001) pp. 108-125

[29] G.G. Joseph; *The Crest of the Peacock—Non European Roots of Mathematics*, Penguin Books, (1992) pp. 108-125

[30] S.C. Kak; Astronomy of Vedic Altars; Vistas in Astronomy, 36, (1993) pp.117-40

[31] S.C. Kak; The sign for zero; The Mankind Quarterly, 30(3) (1990) pp.199-204

[32] S.C. Kak; Indus Writing; The Mankind Quarterly, 30 (1989) pp.113-118

[33] S.C. Kak; Indus and Brahmi, Further Connections; Cryptologica, 14 (1990) pp.169-183

[34] Robert Kaplan; *The Nothing That Is : A Natural History of Zero*, Oxford University Press, (2000).

[35] F.E.J. Linton;*Shedding Some Localic and Linguistic Light on the Tetralemma Conundrums*,Contributions to the History of Indian Mathematics, Emch,Sridharan,Srinivas(edt.), Hindusthan Book Agency, (2005) pp.63-73

[36] R.N. Mukherjee; *Zero*, Encyclopaedia of Classical Indian Sciences, H. Selin, R. Narasimha (edt.), Universities Press, (2007) pp. 441-444

[37] R. Mukherjee; *Discovery of zero and its impact on Indian Mathematics*, Calcutta (1991)

[38] A. Mukhopadhyay; *Impact of Indian Philosophy on Bhāskarīya Concept of Zero*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.197

[39] V.K. Murty; *A Brief History of Indian Mathematics* , *Prabuddha Bhārata*, (Sep. 2007) pp. 517-523,

[40] J. Needham and Wang Ling *Science and Civilization in China* , Vol.II, Cambridge, (1959) pp.10

[41] O. Neugebauer; *The exact Sciences in Anitiquity*, Princeton, (1952)

[42] B. Van Nooten; *Binary numbers in Indian Antiquity*, Journal of Indian Philosophy, 21, pp.31-50, (1993)

[43] J.J. O'Connor and E.F.Robertson;*An Overview of Indian Mathematics*, http://www-history.mcs.st-andrews.ac.uk/HistTopics/Indian_ mathematics.html(November, 2000)

[44] M.D. Pandit; *Zero in Pāṅini's*, University of Poona (1990).

[45] M.D. Pandit; *Reflections on Pāṅinian Zero*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.101-125

[46] S.A. Paramhans; *Philosophical and Mathematical Implications of Zero in Indian Culture*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.206-210

[47] Ian Pearce; *Indian Mathematics: Redressing the Balance*, http://www-history.mcs.st-andrews.ac.uk/history/Projects/Pearce/index.html(May, 2002)

[48] Kim Plofker; *Mathematics in India*; Princeton Universities Press, (2009)

[49] David Pingree; *Zero and the Symbol for zero in early sexagesimal and Decimal Place-Value Systems*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.137-141

[50] C.K. Raju; *The mathematical Epistemology of Śūnya*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.170-183

[51] P. Ray and S.N. Sen(edt.); *The Cultural Heritage of India*, Vol. VI (Science and Technology), The R.K. Mission Institute of Culture, Kolkata (1986).

[52] K.V.Sarma; *Word and Alphabetic numeral Systems in India*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.59

[53] S.A.S. Sarma; *Vedic numerical System including Śūnya*, The Concept of Śūnya, A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp.32

[54] S.R. Sarma; *Śūnya, mathematical Aspect*, *Kalātattvakośa* : A Lexicon of Fundamental Concepts of the Indian Arts, Bettina Baumer (edt.) Vol. 2, pp.400-411, (1992)

[55] S.R. Sarma; *Śūnya in Piṅgala 's Chandaḥsūtra*, The Concept of *Śūnya*, A.K.Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp.126-136

[56] Asoke Chatterjee Sastri; *Piṅgalacchandaḥsūtra : A study*, University of Calcutta, (1987)

[57] R. Sathyanarayana; *Śūnya : A Holistic View*, The Concept of , A.K.Bag and S.R.Sarma (edt.), Aryan Books International, (2003) pp. 264-287

[58] A. Seidenberg; *Origin of Mathematics*, *Archiive for Hist. of Exact Sciences*, 18(4) (1978), pp.301-342

[59] Charles Seife; *Zero— The Biography of a dangerous Idea*, Penguin Books, (2000)

[60] R.S. Sarma (edt.); *Brāhmasphuṭa Siddhānta*, Indian Institute of Astronomical and Sanskrit Research, New Delhi (1966) pp.243

[61] R. Sridharan; *Sanskrit Prosody, Piṅgala Sūtras and Binary Arithmetic*, Contributions to the History of Indian Mathematics, Emch,Sridharan,Srinivas(edt.), Hindusthan Book Agency, (2005)pp. 33-62

[62] Jīvānanda Vidyāsāgara; *Piṅgalacchandaḥsūtra with the* Mṛtasañjıvinī *commentary by* Halāyudha *and a further commentary* , Calcutta, (1928)